Reduction of the Low-Frequency Unsteady Lifting-Surface Problem

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Nomenclature

ΔC_p	=	modified pressure jump
Δc_p	=	unsteady pressure jump, referred to $(\rho_\infty/2)U_\infty^2$
i	=	$(-1)^{\frac{1}{2}}$, imaginary unit
k	=	$\omega l/U_\infty$, reduced frequency
l	=	reference length
M_∞	=	freestream Mach number
t	=	nondimensional time, referred to l/U_∞
U_∞	=	freestream velocity
W	=	modified upwash
w	=	unsteady upwash
x,y,z	=	nondimensional Cartesian coordinates
ω	=	circular frequency
Φ	=	modified velocity potential
ϕ	=	velocity potential, referred to $U_\infty l$
ρ_∞	=	freestream density

Subscripts and superscripts

qs	=	quasisteady
s	=	steady
*	=	complex amplitude
(0),(1)	=	zeroth- and first-order term of frequency expansion

Introduction

AERODYNAMIC loads on slowly oscillating lifting surfaces can be computed using a low-frequency approximation to unsteady aerodynamic theory. This approximate theory is characterized by the assumption that the reduced frequency of the oscillation is small compared to unity. It is very useful as a basis for estimating the damping characteristics of large airplanes. The purpose of this Note is to show that, in subsonic and supersonic flow, the low-frequency problem can be reduced to a sequence of steady lifting-surface problems of the familiar type which are solved by existing methods.

Several reductions of the unsteady problem are known for supersonic flow, but none provides a complete reduction to the familiar steady flow problem valid for wings of arbitrary planform. Reductions for restricted types of wings were developed by Sauer,[1] Magnaradze,[2] Stewartson,[3] and Gardner.[4] Miles[5] has published a reduction formula applicable to arbitrary wings, but the equivalent steady flow problems are of the familiar type only for wings with supersonic trailing edges. Miles[6] has also published a detailed survey of low-frequency theories. A reduction formula for subsonic flow has not appeared in the literature.

Boundary Value Problem

Consider a thin wing in a uniform subsonic or supersonic inviscid flow, as shown in Fig. 1, which is oscillating normal to its projection in the x,y plane. Since harmonic oscillations are assumed, the problem may be described in terms of the complex amplitude of the velocity potential, defined by

$$\phi(x,y,z,t) = \phi^*(x,y,z)e^{ikt} \tag{1}$$

The linear differential equation governing ϕ^* in subsonic and supersonic flow is

$$(1 - M_\infty^2)\phi_{xx}^* + \phi_{yy}^* + \phi_{zz}^* - 2ikM_\infty^2\phi_x^* + k^2M_\infty^2\phi^* = 0 \tag{2}$$†

Received June 18, 1970; revision received July 10, 1970. This research was carried out under NASA-Ames Contract NAS 2-5006. The author wishes to express his appreciation to A. R. Dusto who suggested the problem and offered helpful advice and criticism.

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† Partial differentiation is indicated by subscripts.

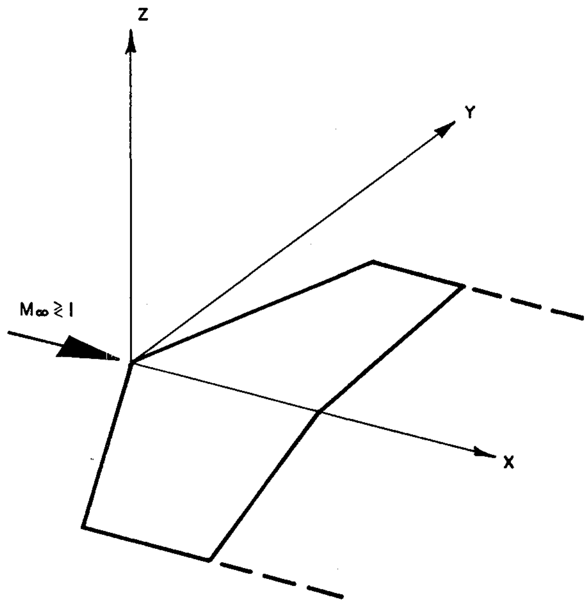

Fig. 1 Coordinate system.

The boundary condition of tangential flow at the lifting surface reads

$$\phi_z^*|_{z=0} = w^*(x,y) \tag{3}$$

where w^* is the given upwash. Across the wake, the pressure must be continuous. This is stated as

$$(\phi_x^* + ik\phi^*)|_{z=0} = 0 \tag{4}$$

The perturbation potential $\phi^*(x,y,z)$ vanishes in subsonic flow for (x,y,z) at infinity and in supersonic flow for (x,y,z) ahead of the Mach cone from the foremost point of the wing. The familiar steady boundary value problem, referred to in the introduction, is obtained from Eqs. (2–4) for $k = 0$.

Reduction to Steady Flow

The analysis is restricted to reduced frequencies that are small compared to unity. The most direct reduction is performed by defining a modified velocity potential Φ and a corresponding upwash W as

$$\Phi_x = (\phi_x^* + ik\phi^*)e^{ik[M_\infty^2/(M_\infty^2-1)]x} \tag{5}$$

$$W_x = (w_x^* + ikw^*)e^{ik[M_\infty^2/(M_\infty^2-1)]x} \tag{6}$$

These forms can be deduced by applying a Lorentz transformation to the unsteady wing problem formulated in space fixed coordinates for the pressure or acceleration potential.

The modified potential Φ is a solution of

$$(1 - M_\infty^2)\Phi_{xx} + \Phi_{yy} + \Phi_{zz} + k^2\frac{M_\infty^2}{1 - M_\infty^2}\Phi = 0 \tag{7}$$

$$\Phi_z|_{z=0} = W(x,y) \text{ on the surface} \tag{8}$$

$$\Phi_x|_{z=0} = 0 \text{ at the wake} \tag{9}$$

with the additional condition of vanishing perturbations at infinity. Expanding Φ and W to first order in frequency, $\Phi = \Phi^{(0)} + ik\Phi^{(1)}$, $W = W^{(0)} + ikW^{(1)}$, and neglecting higher-order terms in k results in

$$(1 - M_\infty^2)\Phi_{xx}^{(\ )} + \Phi_{yy}^{(\ )} + \Phi_{zz}^{(\ )} = 0 \tag{10}$$

$$\Phi_z^{(\ )}|_{z=0} = W^{(\ )}(x,y) \text{ on the surface} \tag{11}$$

$$\Phi_x^{(\ )}|_{z=0} = 0 \text{ at the wake} \tag{12}$$

These equations describe two boundary value problems of the familiar steady type for the potentials $\Phi^{(0)}$ and $\Phi^{(1)}$. The upwashes $W^{(0)}$ and $W^{(1)}$ are different and have to be expressed in terms of the given unsteady upwash w^*. Expanding Eq. (6) for low frequencies, in particular $w^* = w^{(0)} + ikw^{(1)}$, yields

$$W^{(0)} = w^{(0)} \tag{13}$$

$$W^{(1)} = w^{(1)} + \frac{M_\infty^2}{M_\infty^2 - 1} xw^{(0)} + \frac{1}{1 - M_\infty^2}\int_{-\infty}^{x} w^{(0)}d\xi \tag{14}$$

It remains to express the unsteady potential ϕ^* in terms of steady potentials $\Phi^{(0)}$ and $\Phi^{(1)}$. A first order in frequency expansion of Eq. (5), where ϕ^* is expanded as $\phi^* = \phi^{(0)} + ik\phi^{(1)}$, results in

$$\phi^{(0)} = \Phi^{(0)} \tag{15}$$

$$\phi^{(1)} = \Phi^{(1)} + \frac{M_\infty^2}{1 - M_\infty^2} x\Phi^{(0)} + \frac{1}{M_\infty^2 - 1}\int_{-\infty}^{x} \Phi^{(0)}d\xi \tag{16}$$

The desired reduction formula for the unsteady velocity potential is therefore

$$\phi^* = \Phi^{(0)} + ik\left(\Phi^{(1)} + \frac{M_\infty^2}{1 - M_\infty^2} x\, \Phi^{(0)} + \frac{1}{M_\infty^2 - 1} \times \int_{-\infty}^{x} \Phi^{(0)}d\xi\right) \tag{17}$$

The author[7] showed that a low-frequency expansion of ϕ^* of this type is valid for finite wings in subsonic flow and arbitrary wings in supersonic flow. The frequency is strongly restricted beyond the low-frequency assumption $k \ll 1$ by the Mach number. $\Phi^{(0)}$ and $\Phi^{(1)}$ are potentials of wings in steady flight with upwashes given by Eqs. (13) and (14). From Eq. (17) follows the unsteady pressure amplitude in terms of steady pressures as

$$\Delta c_p^* = \Delta C_p^{(0)} + ik\left(\Delta C_p^{(1)} + \frac{M_\infty^2}{1 - M_\infty^2} x\, \Delta C_p^{(0)}\right) \tag{18}$$

where $\Delta C_p^{(0)}$ and $\Delta C_p^{(1)}$ are zeroth and first-order terms of the modified pressure jump $\Delta C_p = \Phi_x$.

Miles' Reduction Formula

Equation (5) can be identified as the application of a well-known modification to the pressure or acceleration potential $\phi_x^* + ik\phi^*$. The same modification applied to the velocity potential ϕ^*, as done by Miles,[6] does not yield a complete reduction to steady flow. Miles' supersonic formula reads

$$\phi^*(x,y,z) = \phi_s(x,y,z;w^{(0)}) + ik\Big\{\phi_s(x,y,z;w^{(1)}) + \frac{M_\infty^2}{M_\infty^2 - 1}[\phi_s(x,y,z;xw^{(0)}) - x\phi_s(x,y,z;w^{(0)})] - \frac{1}{M_\infty^2 - 1}\int_{-\infty}^{x}[\phi_{qs}(\xi,y,z;w^{(0)}) - \phi_s(\xi,y,z;w^{(0)})]d\xi\Big\} \tag{19}$$

where the unsteady upwash is already expanded to first order in frequency. ϕ_s is a steady potential. ϕ_{qs} is quasisteady, i.e., it is a solution of a steady boundary value problem in which the wake condition is replaced by $\phi_{qs}|_{z=0} = 0$. The notation $\phi_s(x,y,z;w^{(0)})$ indicates that ϕ_s is the potential of a wing with upwash $w^{(0)}$.

Comparing Eq. (19) with the new reduction formula (17) shows that the quasi-steady potential in Miles' formula can be replaced by a steady potential.

$$\int_{-\infty}^{x} \phi_{qs}(\xi,y,z;w^{(0)})d\xi = \phi_s(x,y,z;\int_{-\infty}^{x} w^{(0)}d\xi) \tag{20}$$

The steady potential ϕ_s satisfies the boundary condition on the lifting surface

$$\phi_{s_z}|_{z=0} = \int_{-\infty}^{x} w^{(0)}(\xi,y)d\xi \tag{21}$$

Conclusions

The low-frequency problem in unsteady aerodynamics has been reduced completely to a pair of familiar steady wing problems. The reduction formula for the velocity potential of slowly oscillating wings at subsonic and supersonic speeds is given by Eq. (17). For known unsteady upwash, the upwashes of the equivalent steady wings are obtained from Eqs. (13) and (14). The reduction formula can be used to apply existing steady aerodynamic programs for planar or nonplanar configurations to the problem of calculating dynamic stability derivatives. For this first order in frequency approximate theory, the stability derivatives do not depend on the reduced frequency. Hence, they may be used to compute stability characteristics by the usual solution of an eigenvalue problem.

References

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Technical Comment

Erratum: "Sonic Boom Minimization Schemes"

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[J. Aircraft, **7**, 280–281 (1970)]

UNFORTUNATELY, the following transcription errors escaped the proofreading process.[1] Equation (2) should read

$$\dot{m}(x) = \rho_\infty U_\infty \int_0^x [1 + \tfrac{1}{2}M_\infty^2 C_{p0}(t)]\Delta A'(t)dt$$

Equation (3c) should read

$$\frac{\gamma}{\gamma - 1}\rho u A\left(\frac{1}{\rho}\frac{dP}{dx} - \frac{P}{\rho^2}\frac{d\rho}{dx} + \frac{\gamma - 1}{\gamma}u\frac{du}{dx}\right) = \dot{q}(x)$$

Reference

[1] Siegelman, D., "Sonic Boom Minimization Schemes," *Journal of Aircraft*, Vol. 7, No. 3, May–June 1970, pp. 280–281.